

AN ENTROPIC CHARACTERIZATION OF LONG MEMORY STATIONARY PROCESS

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ABSTRACT. Long memory or long range dependency is an important phenomenon that may arise in the analysis of time series or spatial data. Most of the definitions of long memory of a stationary process $X = \{X_1, X_2, \dots\}$ are based on the second-order properties of the process. The excess entropy of a stationary process is the summation of redundancies which relates to the rate of convergence of the conditional entropy $H(X_n|X_{n-1}, \dots, X_1)$ to the entropy rate. It is proved that the excess entropy is identical to the mutual information between the past and the future when the entropy $H(X_1)$ is finite. We suggest the definition that a stationary process is long memory if the excess entropy is infinite. Since the definition of excess entropy of a stationary process requires very weak moment condition on the distribution of the process, it can be applied to processes whose distributions without bounded second moment. A significant property of excess entropy is that it is invariant under invertible transformation, which enables us to know the excess entropy of a stationary process from the excess entropy of other process. For stationary Gaussian process, the excess entropy characterization of long memory relates to popular characterization well. It is proved that the excess entropy of fractional Gaussian noise is infinite if the Hurst parameter $H \in (1/2, 1)$.

Key words: Long memory, excess entropy, mutual information, stationary process, fractional Gaussian noise

1. INTRODUCTION

Long memory and long range dependence are synonymous notions, are the phenomenon that may arise in the analysis of time series or spatial data, and are very important. This importance can be judged, for example, by the very large number of publications having one of these notions in the title, in areas such as finance [12], econometrics [15], internet modeling [10], hydrology [14], climate studies [19] or linguistics [1].

A stationary process is a sequence of random variables, whose probability law is time invariant. Long Memory phenomenon relates to the rate of decay of statistical dependence of a stationary process, with the implication that this decays more slowly than an exponential decay, typically a power-like decay. Some self-similar processes may exhibit long memory, but not all processes having long memory are self-similar. When definitions are given, they vary from author to author (the econometric survey [8] mentions 11 different definitions). Different definitions of long memory are used for different applications. Most of the definitions of long memory appearing in literature are based on the second-order properties of a stochastic process. Such properties include asymptotic behavior of covariances, spectral density, and variances of partial sums. The reasons for popularity of the second-order properties in this context are both historical and practical: second-order properties are relatively simple conceptually and easy to estimate from the data. The notion

of long memory is discussed from a variety of points of view, and a comprehensive review is given by Samorodnitsky [18].

Concentrating too much on the correlations has a number of drawbacks [18]. Firstly, correlations provide only very limited information about the process if the process is "not very close" to being Gaussian. Secondly, rate of decay of correlations may change significantly after instantaneous one-to-one transformations of the process. Finally, what to do if the variance is infinite?

Whatever the drawbacks of using correlations to measure length of memory in the L^2 case, the whole approach breaks down when the variance is infinite. Some of the proposed ways out in specific situations included computing "correlation-like" numbers, or using instead characteristic functions by studying the rate of convergence to zero of the difference. This approaches have met only with limited success.

The question then arises of whether it is possible to develop a new approach to solve the number of drawbacks. In information theory, the *excess entropy* was developed as intuitive measure of memory stored in a stationary stochastic process, which is related to the mutual information between the infinite past and the infinite future; see Section 2. It has a long history and is widely employed as a measure of correlation and complexity in a variety of fields, from ergodic theory and dynamical systems to neuroscience and linguistics, see Ref. [6] and references therein for a review.

We advocate a different approach to the problem of long memory of stationary process $X = \{X_1, X_2, \dots\}$ by *excess entropy*. We stress that a stationary process is long memory if the excess entropy of X is infinite. Such a characterization of long memory stationary process admits many advantages:

- (1) The definition of excess entropy requires $H(X_1) < +\infty$ rather than the second moment condition $\mathbb{E}X_1^2 < \infty$, so it can be used to detect the long memory behavior of stationary process with heavy tail distribution;
- (2) The excess entropy is invariant under 1-1 transformation (Theorem 2);
- (3) It is closely related to second moment characterization if the stationary process is Gaussian (Theorem 3).

The paper is organized as follows. In section 2 we introduce basic concepts of information theory and show some properties of excess entropy: the excess entropy is identical to the mutual information between the past and the future, and the excess entropy is invariant under 1-1 transformation. In section 3 and section 4, we illustrate how excess entropy relates to covariance when the stationary process is Gaussian, and show that the excess entropy is infinite for the fractional Gaussian noise with Hurst parameter $H \in (1/2, 1)$. We summarize our results in section 5.

2. EXCESS ENTROPY AND MUTUAL INFORMATION

At first, we collect some basic concepts and theorems in information theory from Chapter 8 of the book [5]. The Shannon entropy $H(X)$ of a discrete random variable X taking values $x \in S$ is defined as

$$(2.1) \quad H(X) := -\mathbb{E}[\log P(X)] = -\sum_{x \in S} P(x) \log P(x),$$

where the probability that X takes on the particular value x is written $P(x) \equiv P(X = x)$. The joint entropy $H(X, Y)$ of a pair of discrete random variables

(X, Y) with a joint distribution $p(x, y)$ is defined as

$$H(X, Y) := -\mathbb{E} \log p(X, Y) = -\sum_x \sum_y p(x, y) \log p(x, y).$$

The conditional entropy $H(X|Y)$, which is the entropy of a X conditional on the knowledge of another random variable Y is

$$H(X|Y) := -\mathbb{E} \log p(X|Y) = -\sum_x \sum_y p(x, y) \log p(x|y).$$

The differential entropy $H(X)$ of a continuous random variable X with density $f(x)$ is defined as

$$(2.2) \quad H(X) := -\int_S f(x) \log f(x) dx,$$

where S is the support set of the random variable.

If X, Y have a joint density function $f(x, y)$, the joint entropy $H(X, Y)$ of a pair of random variables (X, Y) is defined as

$$H(X, Y) = -\int \int f(x, y) \log f(x, y) dx dy,$$

and the conditional differential entropy $H(X|Y)$ as

$$H(X|Y) = -\int \int f(x, y) \log f(x|y) dx dy.$$

Since in general $f(x|y) = f(x, y)/f(y)$, we can also write

$$H(X|Y) = H(X, Y) - H(Y).$$

The following chain rule with two random variables is valid for both discrete and continuous random variables [5]

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y).$$

Consider two random variables X and Y with a joint probability density function $f(x, y)$ and marginal probability density functions $f(x)$ and $f(y)$. The mutual information $I(X; Y)$ is the relative entropy between the joint distribution and the product distribution $f(x)f(y)$:

$$I(X; Y) = \int \int f(x, y) \log \frac{f(x, y)}{f(x)f(y)} dx dy.$$

For discrete random variables X and Y ,

$$I(X; Y) = \sum_x \sum_y p(x, y) \log \frac{p(x, y)}{p(x)p(y)}.$$

It follows that

$$I(X; Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X, Y).$$

The joint entropy of a set X_1, X_2, \dots, X_n of random variables with density $f(x_1, x_2, \dots, x_n)$ is defined as

$$H_X(n) := H(X_1, X_2, \dots, X_n) = -\int f(x_1, \dots, x_n) \log f(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

For convenience, $H_X(n)$ is denoted by $H(n)$ if the underlying process X is known, $H(0) = 0$.

A stochastic process $X = \{X_i\}$ is an indexed sequence of random variables. A stochastic process is stationary if the joint probability distribution does not change

when shifted in time. The following Lemma collects some properties of the block entropy sequence $H(n)$ of a stationary process X .

Lemma 2.1. *Let $X = \{X_1, \dots, X_n, \dots\}$ be a stationary process such that the entropy $H(X_1)$ is finite. We have the following*

(1) *The chain rule:*

$$(2.3) \quad H(X_1, X_2, \dots, X_n) = \sum_{i=1}^n H(X_i | X_{i-1} \dots X_1).$$

(2) *The entropy gain $h(n) := H(n) - H(n-1) = H(X_n | X_{n-1}, \dots, X_1)$ is nonnegative;*

(3) *$h(n)$ is nonincreasing and $\lim_{n \rightarrow \infty} h(n) = h_\mu$;*

(4) *$H(n)$ is concave, i.e., for $\lambda \in (0, 1)$,*

$$H(\lambda m + (1 - \lambda)n) \leq \lambda H(m) + (1 - \lambda)H(n)$$

provided that m, n and $\lambda m + (1 - \lambda)n$ are nonnegative integers;

(5) *$H(n)$ is subadditive: for all nonnegative integers m and n ,*

$$H(n + m) \leq H(n) + H(m);$$

Proof:

(1) It follows easily from chain rule in two random variables, see [5].

(2) By the chain rule, we get the first derivative of $H(n)$ is

$$\begin{aligned} h(n) &:= H(n) - H(n-1) \\ &= \sum_{i=1}^n H(X_i | X_{i-1} \dots X_1) - \sum_{i=1}^{n-1} H(X_i | X_{i-1} \dots X_1) \\ &= H(X_n | X_{n-1}, \dots, X_1) \geq 0. \end{aligned}$$

(3) By the stationarity of X , we have

$$\begin{aligned} h(n) - h(n-1) &= H(X_n | X_{n-1}, \dots, X_1) - H(X_{n-1} | X_{n-2}, \dots, X_1) \\ &\leq H(X_n | X_{n-1}, \dots, X_2) - H(X_{n-1} | X_{n-2}, \dots, X_1) \\ &= 0. \end{aligned}$$

(4) By the previous statement, the second order derivative of $H(n)$ is

$$(H(n) - H(n-1)) - (H(n-1) - H(n-2)) = h(n) - h(n-1) \leq 0,$$

which indicates that $H(n)$ is concave.

(5) Since $h(n)$ is nonincreasing, by the chain rule we know that

$$H(n+m) = \sum_{i=1}^{n+m} h(i) = \sum_{i=1}^n h(i) + \sum_{i=n+1}^{n+m} h(i) \leq H(n) + \sum_{i=1}^m h(i) = H(n) + H(m).$$

□

The *entropy rate* of a stochastic process $X = \{X_i\}$, $X_i \in R$, is defined to be

$$h_\mu = h_\mu(X) := \lim_{n \rightarrow \infty} \frac{1}{n} H(n) = \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1, X_2, \dots, X_n)$$

if the limit exists. For a stationary stochastic process, by Lemma 1, the above limit always exists because the sequence $H(n)$ is sub-additive.

On the other hand, by Lemma 1, $h(n)$ is nonnegative and nonincreasing, which will approach to the same limit h_μ because

$$h_\mu = \lim_{n \rightarrow \infty} \frac{H(n)}{n} = \lim_{n \rightarrow \infty} \frac{H(n) - H(n-1)}{n - (n-1)} = \lim_{n \rightarrow \infty} h(n).$$

The existence of entropy rate for stationary process is proved by Shannon [1]. A significant property of entropy rate is the AEP (asymptotic equipartition property), also known as the Shannon-McMillan-Breiman theorem: If h_μ is the entropy rate of a finite-valued stationary ergodic process $\{X_n\}$, then

$$\frac{1}{n} \log p(X_0, \dots, X_n) \rightarrow h_\mu$$

with probability 1. The entropy rate h_μ quantifies the irreducible randomness in sequences produced by a source: the randomness that remains after the correlations and structures in longer and longer sequence blocks are taken into account. It is known as the metric entropy in ergodic theory.

For stationary process X , by the definition of entropy rate,

$$H(n) \sim nh_\mu \quad \text{as } n \rightarrow \infty.$$

However, knowing the value h_μ indicates nothing about how $H(n)/n$ approaches this limit. Moreover, there may be sublinear terms in $H(n)$. For example, one may have $H(n) \sim nh_\mu + c$ or $H(n) \sim nh_\mu + \log n$. The sublinear terms in $H(n)$ and the manner in which $H(n)$ converges to its asymptotic form may reveal important structural properties about a process.

The *excess entropy* of a stationary process X is defined as:

$$E = \sum_{n=1}^{\infty} (h(n) - h_\mu).$$

$h(n) - h_\mu$ is referred as per-symbol redundancy $r(n)$, because it tells us how much additional information must be gained about the process in order to reveal the actual per-symbol uncertainty h_μ . In other words, the excess entropy E is the summation of per-symbol redundancy [6]. Substitute $h(n) = (H(n) - H(n-1))$ into the definition of the excess entropy, we know that

$$E = \lim_{n \rightarrow \infty} (H(n) - nh_\mu).$$

If the excess entropy is finite, we obtain

$$H(n) \approx nh_\mu + E$$

as $n \rightarrow \infty$.

Observe that $H(n)/n \rightarrow h_\mu$, one may ask if $F = \sum_{n=1}^{\infty} (\frac{H(n)}{n} - h_\mu)$ can be used to describe the long memory of stationary process like excess entropy. It is not the case. In fact,

$$\begin{aligned} F &= \sum_{n=1}^{\infty} \left(\frac{H(n)}{n} - h_\mu \right) \\ &= \sum_{n=1}^{\infty} \frac{(H(n) - H(n-1)) + \dots + (H(1) - H(0)) - nh_\mu}{n} \\ &= \sum_{n=1}^{\infty} \frac{(h(1) - h_\mu) + (h(2) - h_\mu) + \dots + (h(n) - h_\mu)}{n} \\ &\geq (h(1) - h_\mu) \sum_{n=1}^{\infty} \frac{1}{n}, \end{aligned}$$

which is divergent for all stationary process with $H(1) > h_\mu$. Notice that $H(1) = h_\mu$ if and only if the stationary process is independent, so $F = \infty$ unless the process is *i.i.d.*.

The mutual information $I_{p-f}(n) := I(X_1 \cdots X_n; X_{n+1} \cdots X_{2n})$ is the information between the history and future with length n , and the mutual information between past and future is the limit $I_{p-f} = \lim_{n \rightarrow \infty} I_{p-f}(n)$.

Theorem 2.2. *Let $X = \{X_1, \dots, X_n, \dots\}$ be a stationary process such that the entropy $H(X_1)$ is finite. Then the excess entropy and the mutual information are identical:*

$$E = I_{p-f}.$$

Proof: Denote

$$E_n := \sum_{k=1}^n (H(k) - H(k-1) - h_\mu) = \sum_{k=1}^n (h(k) - h_\mu) = H(n) - nh_\mu.$$

$$\begin{aligned} I_{p-f}(n) &:= I(X_1, \dots, X_n; X_{n+1}, \dots, X_{2n}) \\ &= H(X_1, \dots, X_n) + H(X_{n+1}, \dots, X_{2n}) - H(X_1, \dots, X_{2n}) \\ &= 2H(n) - H(2n). \end{aligned}$$

Put

$$\begin{aligned} D_n &:= H(n) - nh(n) = nH(n-1) - (n-1)H(n) \\ &= n(n-1) \left(\frac{H(n-1)}{n-1} - \frac{H(n)}{n} \right) \geq 0. \end{aligned}$$

Denote $a_n = h(n) - h_\mu$, by the definition of entropy rate, we know that $a_n \geq 0$, $a_n \geq a_{n+1}$ for positive integer n , and $\lim_{n \rightarrow \infty} a_n = 0$. Furthermore, we have

$$(2.4) \quad E_n = \sum_{k=1}^n a_k, \quad I_{p-f}(n) = \sum_{k=1}^n a_k - \sum_{k=n+1}^{2n} a_k, \quad D_n = \sum_{k=1}^n a_k - na_n.$$

We conclude that

$$(2.5) \quad E_n \geq I_{p-f}(n) \geq D_n.$$

In fact, the first inequality follows from $\{a_n\}$ is nonnegative, and the second inequality follows from $\{a_n\}$ is nonincreasing.

Since E_n is the partial summation of the nonnegative series $\{a_n\}$, E_n is nondecreasing. Notice that

$$\begin{aligned} I_{p-f}(n+1) - I_{p-f}(n) &:= \left(\sum_{k=1}^{n+1} a_k - \sum_{k=n+2}^{2n+2} a_k \right) - \left(\sum_{k=1}^n a_k - \sum_{k=n+1}^{2n} a_k \right) \\ &= 2a_{n+1} - (a_{2n+1} + a_{2n+2}) \\ &= (a_{n+1} - a_{2n+1}) + (a_{n+1} - a_{2n+2}) \geq 0, \end{aligned}$$

and

$$\begin{aligned} D(n+1) - D_n &:= \left(\sum_{k=1}^{n+1} a_k - (n+1)a_{n+1} \right) - \left(\sum_{k=1}^n a_k - na_n \right) \\ &= n(a_n - a_{n+1}) \geq 0, \end{aligned}$$

both $I_{p-f}(n)$ and D_n are nondecreasing. It follows that the following three limits exists:

$$(2.6) \quad \lim_{n \rightarrow \infty} E_n = E, \quad \lim_{n \rightarrow \infty} I_{p-f}(n) = I_{p-f}, \quad \lim_{n \rightarrow \infty} D_n = D.$$

Claim: E_n is convergent if and only if D_n is convergent, and they have the same limit when they are convergent.

If $E_n = \sum_{k=1}^n a_k$ is convergent, then the series $\sum_{k=1}^{\infty} 2^k a_{2^k}$ is convergent. So $2^k a_{2^k} \rightarrow 0$ as $k \rightarrow \infty$. Observe that for $2^k < n \leq 2^{k+1}$, $na_n \leq 2^{k+1} a_{2^k}$, we get $\lim_{n \rightarrow \infty} na_n = 0$. Hence

$$\lim_{n \rightarrow \infty} D_n = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n a_k - na_n \right) = \lim_{n \rightarrow \infty} E_n - \lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} E_n.$$

Suppose D_n is convergent. Notice that

$$D_n = \sum_{k=1}^n a_k - na_n = \sum_{k=1}^{n-1} k(a_k - a_{k+1}),$$

it follows that $\forall \varepsilon > 0, \exists N > 0, \forall n, s \in \mathbb{N}, n > N$,

$$\sum_{k=n}^{n+s} k(a_k - a_{k+1}) \leq \frac{\varepsilon}{2}.$$

As a result,

$$0 \leq n(a_n - a_{n+s+1}) = \sum_{k=n}^{n+s} n(a_k - a_{k+1}) \leq \sum_{k=n}^{n+s} k(a_k - a_{k+1}) \leq \frac{\varepsilon}{2}.$$

Hence,

$$na_n \leq \frac{\varepsilon}{2} + na_{n+s+1}.$$

On the other hand, $\lim_{s \rightarrow \infty} a_{n+s+1} = 0$ implies that $\forall n > N, \exists S \in \mathbb{N}, \forall s > S$,

$$0 < a_{n+s+1} < \frac{\varepsilon}{2n}.$$

We obtain for $n > N$,

$$0 \leq na_n \leq \frac{\varepsilon}{2} + na_{n+s+1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which indicates $\lim_{n \rightarrow \infty} na_n = 0$.

Therefore,

$$\lim_{n \rightarrow \infty} E_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{n \rightarrow \infty} D_n + \lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} D_n.$$

The claim is proved.

By (2. 5) and the claim, we conclude that

$$E = I_{p-f} = D.$$

□

Remarks:

- (1) The equality $E = I_{p-f}$ is claimed in [6] for discrete stationary process, an heuristic "proof" was also given. The proof is too simple to be complete.
- (2) The definition of excess entropy E is depend on the entropy rate h_μ . The equation $E = I_{p-f} = D$ provide two series to approximate the excess entropy $\{2H(n) - H(2n)\}_n$ and $\{nH(n-1) - (n-1)H(n)\}_n$, which enable us to obtain lower bound of the excess entropy E without knowing the entropy rate h_μ .

In what follows we show that the excess entropy is invariant under 1-1 transformation.

Lemma 2.3. *Let Z be a random variable with finite entropy, T is a deterministic transformation, then $H(Z) \geq H(T(Z))$. Moreover, if T is an invertible transformation, we have $H(Z) = H(T(Z))$.*

Proof. By the chain rule of two random variables, we have $H(Z, T(Z)) = H(Z) + H(T(Z)|Z) = H(T(Z)) + H(Z|T(Z))$. Note that $H(T(Z)|Z) = 0$ and $H(Z|T(Z)) \geq 0$, we know that $H(Z) \geq H(T(Z))$. Furthermore, if T is invertible, we have

$$H(Z) \geq H(T(Z)) \geq H(T^{-1}(T(Z))) = H(Z),$$

which implies $H(Z) = H(T(Z))$. \square

Theorem 2.4. *Let $X = \{X_1, X_2, \dots\}$ be a stationary stochastic process, T be an invertible transformation, then the excess entropy of the process $Y := T(X) = \{T(X_1), T(X_2), \dots\}$ equals to the excess entropy of X .*

Proof. Put $Z_n = (X_1, \dots, X_n)$, $T(Z_n) = (T(X_1), \dots, T(X_n))$. Since T is an invertible transformation, by lemma 2, we have $H(Z_n) = H(T(Z_n))$, which implies $H_X(n) = H_Y(n)$ for all $n \geq 0$. As a result, the stationary processes X and Y admit the same the excess entropy. \square

To characterize the long memory by excess entropy, a new definition is suggested as follows.

Definition 2.5. *A stationary process is long memory if its excess entropy is infinite.*

As examples, the excess entropy for two simple short memory processes are as follows.

For *i.i.d.* case, the entropy rate $h_\mu = H(1)$ because $H(n) = nH(1)$, the excess entropy $E = 0$.

For a Markov chain X defined on a countable number of states, given the transition matrix P_{ij} , the entropy rate of X is given by:

$$h_\mu(X) = - \sum_{ij} \mu_i P_{ij} \log P_{ij}$$

where μ_i is the stationary distribution of the chain. By the Markovian property, the excess entropy of the Markov chain is:

$$E(X) = (H(1) - h_\mu) + (H(2) - H(1) - h_\mu) = - \sum_i u_i \log u_i + \sum_{ij} \mu_i P_{ij} \log P_{ij}.$$

3. STATIONARY GAUSSIAN PROCESS

For a stationary Gaussian stochastic process, we have

$$H(X_1, X_2, \dots, X_n) = \frac{1}{2} \log(2\pi e)^n |K^{(n)}|,$$

where the covariance matrix $K^{(n)}$ is Toeplitz with entries $r(0), r(1), \dots, r(n-1)$ along the top row, i.e., $K_{ij}^{(n)} = r(i-j) = E(X_i - EX_i)(X_j - EX_j)$. It is useful to consider a stochastic process in the frequency domain. A stationary zero mean Gaussian random process is completely described by its mean correlation function $r_{k,j} = r_{k-j}$ or, equivalently, by its power spectral density function f , the Fourier transform of the covariance function:

$$f(\lambda) = \sum_{n=-\infty}^{\infty} r_n \exp(in\lambda),$$

$$\gamma_k = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) \exp(-i\lambda k) d\lambda.$$

Indeed, Kolmogorov showed that the entropy rate of a stationary Gaussian stochastic process can be expressed as

$$h_\mu(X) = \frac{1}{2} \log(2\pi e) + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log f(\lambda) d\lambda,$$

where $S(\lambda)$ is the power spectral density of the stationary Gaussian process X . On the other hand,

$$h_\mu(X) = \lim_{n \rightarrow \infty} (H(n) - H(n-1)) = \frac{1}{2} \lim_{n \rightarrow \infty} \log(2\pi e) \frac{|K^{(n)}|}{|K^{(n-1)}|}.$$

Let X be a Gaussian stationary process with spectral density $f(\lambda)$. Put

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) e^{-k\lambda} d\lambda,$$

for any integer k if it is well defined, and we refer to them as cepstrum coefficients [11].

Li proved the following theorem (Theorem 1 in [11])

Theorem 3.1. *Let $\{X_n\}$ be a Gaussian stationary process [11]*

- (i) *The mutual information between the past and the future I_{p-f} is finite if and only if the cepstrum coefficients satisfy the condition: $\sum_{k=-\infty}^{\infty} kb_k^2 < \infty$. In this case, $I_{p-f} = \frac{1}{2} \sum_{k=-\infty}^{\infty} kb_k^2$.*
- (ii) *If the spectral density $f(\lambda)$ is continuous, and $f(\lambda) > 0$, then I_{p-f} is finite if and only if the autocovariance functions satisfy the condition $\sum_{k=-\infty}^{\infty} k\gamma_k^2 < \infty$.*

Lemma 3.2. *Let $\{c(n)\}$ be a decreasing positive series. Then*

$$\sum_n c(n) < \infty \quad \text{implies} \quad \sum_n nc^2(n) < \infty.$$

Proof: Since $\{c(n)\}$ be a decreasing positive series, $\sum_n c(n)$ is convergent implies that $\{nc(n)\} \rightarrow 0$. So $\{nc(n)\}$ is bounded by a constant M . Therefore,

$$\sum_n nc^2(n) \leq \sum_n Mc(n) = M \sum_n c(n) < \infty.$$

□

In general, we have the following result for Gaussian stationary process.

Theorem 3.3. *Let X be a Gaussian stationary process with decreasing autocorrelation $\{|r(n)|\}$. Suppose the spectral density $f(\lambda)$ is continuous and $f(\lambda) > 0$. Then*

$$\sum_n |r(n)| < \infty \Rightarrow \quad \text{the excess entropy } E < \infty.$$

In other words, X is not long memory in the sense of covariance implies that it is also not long memory in the sense of excess entropy.

The proof of Theorem 3.3 follows immediately from Theorem 3.1 and Lemma 3.2.

Remark: Put $a(n) = \frac{1}{n \log n}$, then $\sum_n a(n) = \infty$, but $\sum_n na^2(n) < \infty$. The converse implication in Lemma 3.2 is not true.

So for process considered in Theorem 3.3, the long memory in the sense of excess entropy is more strict than that in the sense of covariance.

4. FRACTIONAL GAUSSIAN NOISE

Fractional Brownian motion (FBM) with Hurst parameter $H \in (0, 1)$ is a centered continuous-time Gaussian process $B^H(\cdot)$ with covariance function

$$(4.7) \quad \rho(s, t) \equiv \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

for $s, t \geq 0$. B^H reduces to an ordinary Brownian motion for $H = 1/2$. The incremental process of a FBM is a stationary discrete-time process and is called *fractional Gaussian noise*, or FGN. We define of the FGN $X = \{X_k : k = 0, 1, \dots\}$ by the autocovariance function

$$\rho_k \equiv \mathbb{E}X_{n+k}X_n = \frac{1}{2}[|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H}]$$

for $k \in \mathbb{Z}$. It is easy to see that

$$(4.8) \quad r_k \sim H(2H-1)|k|^{-2(1-H)}.$$

as $|k| \rightarrow \infty$. Of course, if $H = 1/2$, then $\rho_k = 0$ for all $k \geq 1$ (a Brownian motion has independent increments). One can conclude that the summability of correlations ($\sum_{k=1}^{\infty} |\rho_k| < +\infty$) holds if $0 < H < 1/2$ and it does not hold if $1/2 < H < 1$. Therefore, a FGN with $H > 1/2$ has become commonly accepted as having long memory, and lack of summability of correlations as a popular definition of long memory (Eq. (4.8)).

The stationarity of the increments of the FBM implies that this is a stationary Gaussian process.

In particularly, we have the following result for FGN.

Theorem 4.1. *The excess entropy of the (discrete) increment process of a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ is infinite, i.e., it is long memory in the sense of excess entropy.*

Proof: The spectral density of the increment of fractional Brownian motion $B^H(t)$ (fractional Gaussian noise) is obtained by Sinai [16]

$$f(\lambda) = 2 \sin(\pi H) \Gamma(2H+1) (1 - \cos \lambda) [|\lambda|^{-2H-1} + A_H(\lambda)],$$

where $\Gamma(\cdot)$ denotes the Gamma function and

$$A_H(\lambda) := \sum_{j=1}^{\infty} \{(2\pi j + \lambda)^{-2H-1} + (2\pi j - \lambda)^{-2H-1}\},$$

for $-\pi \leq \lambda \leq \pi$.

The spectral density $f(\lambda) = C(1 - \cos \lambda) \sum_{j=-\infty}^{\infty} |2\pi j + \lambda|^{-(2H+1)}$ is positive, but is not continuous at $\lambda = 0$ when $H \in (1/2, 1)$ because it is proportional to $|\lambda|^{1-2H}$ near $\lambda = 0$. Since $(1 - \cos \lambda) = \sum_{n \geq 1} (-1)^{n-1} \frac{\lambda^{2n}}{(2n)!}$,

$$\begin{aligned} |\lambda|^{-(2H+1)} (1 - \cos \lambda) &= |\lambda|^{-(2H-1)} \sum_{n \geq 1} (-1)^{n-1} \frac{\lambda^{2(n-1)}}{(2n)!} \\ &:= |\lambda|^{-(2H-1)} (1 + g_1(\lambda))/2. \end{aligned}$$

Now we estimate the excess entropy of FGN via the logarithm of the spectral density. We have

$$\begin{aligned} \log f(\lambda) &= \log C + \log(1 - \cos \lambda) |\lambda|^{-(2H+1)} \\ &\quad + \log(1 + |\lambda|^{(2H+1)} A_H(\lambda)). \end{aligned}$$

$\log f(\lambda)$ is an even function on $[-\pi, \pi]$, i.e., $\log f(\lambda) = \log f(-\lambda)$. For $n \neq 0$, we obtain the following decomposition

$$\begin{aligned}
\gamma(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) \exp(in\lambda) d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda) \cos(n\lambda) d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log[(1 - \cos \lambda)|\lambda|^{-(2H+1)}] d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log(1 + |\lambda|^{-(2H+1)} A_H(\lambda)) d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log[|\lambda|^{-(2H-1)}(1 + g_1(\lambda))/2] d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log(1 + |\lambda|^{-(2H+1)} A_H(\lambda)) d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log |\lambda|^{-(2H-1)} d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log[(1 + g_1(\lambda))/2] d\lambda \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log(1 + |\lambda|^{-(2H+1)} A_H(\lambda)) d\lambda \\
&:= \gamma_1(n) + \gamma_2(n) + \gamma_3(n).
\end{aligned}$$

For $\gamma_1(n)$, we get

$$\begin{aligned}
\gamma_1(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log |\lambda|^{-(2H-1)} d\lambda \\
&= \frac{-(2H-1)}{\pi} \int_0^{\pi} \cos(n\lambda) \log \lambda d\lambda \\
&= \frac{(2H-1)}{\pi n} \int_0^{\pi} \log \lambda d \sin(n\lambda) \\
&= \frac{-(2H-1)}{\pi n} \int_0^{\pi} \frac{\sin(n\lambda)}{\lambda} d\lambda.
\end{aligned}$$

Observe that

$$(4.9) \quad \int_0^{\infty} \frac{\sin(ax)}{x} dx = \begin{cases} \frac{\pi}{2} & a > 0 \\ -\frac{\pi}{2} & a < 0. \end{cases}$$

So $|\gamma_1(n)| \rightarrow \frac{2H-1}{2n}$ as n goes to infinity.

For the twice continuously differentiable function g , the Fourier coefficient of order n behaves like $O(\frac{1}{n^2})$ [17]. Observe that $\log[(1 + g_1(\lambda))/2]$ is twice differentiable, there exists positive constant $M_1 < \infty$

$$\gamma_2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\lambda) \log[(1 + g_1(\lambda))/2] d\lambda \leq M_1/n^2.$$

Now we estimate $\gamma_3(n)$. Denote

$$g_2(x) = \log(1 + |x|^{2H+1} A_H(x)).$$

We have

$$g_2'(x) = \frac{(|x|^{2H+1}A_H(x))'}{1 + |x|^{2H+1}A_H(x)},$$

$$g_2''(x) = \frac{(|x|^{2H+1}A_H(x))''|x|^{2H+1}A_H(x) - ((|x|^{2H+1}A_H(x))')^2}{(1 + |x|^{2H+1}A_H(x))^2},$$

where

$$\begin{aligned} (|x|^{2H+1}A_H(x))' &= (2H+1)|x|^{2H}A_H(x) + |x|^{2H+1}A_H'(x) \\ (|x|^{2H+1}A_H(x))'' &= 2H(2H+1)|x|^{2H-1}A_H(x) \\ &\quad + (2H+1)|x|^{2H}A_H'(x) \\ &\quad + |x|^{2H+1}A_H''(x) \end{aligned}$$

$$\begin{aligned} A_H'(x) &= \sum_{j=1}^{\infty} \left(\frac{-(2H+1)}{(2\pi j + x)^{2H+2}} + \frac{(2H+1)}{(2\pi j - x)^{2H+2}} \right) \\ A_H''(x) &= \sum_{j=1}^{\infty} \left(\frac{(2H+1)(2H+2)}{(2\pi j + x)^{2H+3}} + \frac{(2H+1)(2H+2)}{(2\pi j - x)^{2H+3}} \right). \end{aligned}$$

Since $|x|^{2H-1}, |x|^{2H}, |x|^{2H+1}, A_H(x), A_H'(x), A_H''(x)$ are continuous functions on $[-\pi, \pi]$, $g_2''(x)$ are also continuous functions on $[-\pi, \pi]$. We conclude that

$$|\gamma_3(n)| \leq \frac{M}{n^2}$$

for some positive constant $M_2 < \infty$.

Combine the estimations of $\gamma_1(n)$, $\gamma_2(n)$, $\gamma_3(n)$,

$$\lim_{n \rightarrow \infty} \frac{|\gamma(n)|}{n} = \lim_{n \rightarrow \infty} \frac{|\gamma(n) + \gamma_2(n) + \gamma_3(n)|}{n} = (2H+1)/2.$$

It follows that there exists positive integer N_0 such that $|\gamma(n)| \geq \frac{(2H+1)}{4n}$ for $n > N_0$.

Hence, by Theorem 2, the excess entropy

$$\begin{aligned} E &= \sum_{n=1}^{\infty} n|\gamma(n)|^2 > \frac{2H+1}{4} \sum_{n>N_0} n|\gamma(n)|^2 \\ &\geq \frac{2H+1}{4} \sum_{n>N_0} n \frac{1}{n^2} = \infty. \end{aligned}$$

□

5. CONCLUSION

The finiteness or infiniteness of excess entropy can be regarded as a sign between the short memory and long memory stationary processes. The definition of excess entropy of a stationary process requires very weak moment condition on the distribution of the process, and can be applied to processes whose distributions without bounded second moment. The most significant property of excess entropy is that it is invariant under invertible transformation. The invariance under invertible transformation enables us to know the excess entropy of a stationary process from the excess entropy of other process. Since conditional entropy can capture the dependence between random variables well, excess entropy is relevant to capture

the dependence of a stationary process whose distribution far from Gaussian distribution. For stationary Gaussian process, the excess entropy characterization of long memory related to popular characterization neatly. The challenge is to develop pertinent methods and algorithms to estimate or approximate the excess entropy of typical stationary process or sequential data.

ACKNOWLEDGMENTS

We would like to thank Professor Yimin Xiao and Professor Yaozhong Hu for useful discussions and suggestions. This work was supported by National Natural Science Foundation of China (grant no. 11171101), and the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry (2015).

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